

ON SOME NONLINEAR INTEGRAL INEQUALITIES ON TIME SCALES

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ABSTRACT. In this paper we study some nonlinear Pachpatte type integral inequalities on time scales by using a Bihari type inequality. Our results unify some continuous inequalities and their corresponding discrete analogues, and extend these inequalities to dynamic inequalities on time scales. Furthermore, we give some examples concerning our results.

1. Introduction

The Gronwall inequalities play a very important role in the study of the qualitative theory of differential and integral equations. Furthermore, they can be widely used to investigate stability properties for solutions of differential and difference equations. See [1, 5, 7, 13, 17, 18, 21, 25] for differential inequalities and difference inequalities.

Pachpatte [23, 24] obtained some general versions of Gronwall-Bellman inequality [6, 15]. Oguntuase [22] established some generalizations of the inequalities obtained in [23]. However, there were some defects in the proofs of Theorems 2.1 and 2.7 in [22]. Choi et al. [10] improved the results of [22] and gave an application to boundedness of the solutions of nonlinear integro-differential equations.

The theory of time scales (closed subsets of \mathbb{R}) was initiated by Hilger [16] in his Ph. D. thesis in 1988 in order to unify the theories of differential equations and of difference equations and in order to extend those theories to other kinds of the so-called “dynamic equations”. The

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two main features of the calculus on time scales are unification and extension of continuous and discrete analysis. For example, a few papers [2, 26, 20, 28, 3, 11] have studied the theory of integral inequalities on time scales.

In this paper, using a Bihari type inequality on time scales, we establish some nonlinear Pachpatte type integral inequalities on time scales, which provide explicit bounds on unknown functions. Our results unify some differential inequalities in [10] and their corresponding discrete analogues in [12] and extend these inequalities to dynamic inequalities on time scales. Also, the results extend some previous some Pachpatte type inequalities on time scales [19] and provide explicit bounds for some nonlinear integral inequalities in [14]. Furthermore, we give some applications of our results.

2. Main results

We refer the reader to Ref. [8] for all the basic definitions and results on time scales necessary to this work (e.g. delta differentiability, rd-continuity, exponential function and its properties). Let \mathbb{T} be an arbitrary time scale. $C_{rd}(\mathbb{T}, \mathbb{R})$ denotes the set of all rd-continuous functions, $\mathcal{R}(\mathbb{T}, \mathbb{R})$ denotes the set of all regressive and rd-continuous functions, $\mathcal{R}^+ = \{p \in \mathcal{R}(\mathbb{T}, \mathbb{R}) : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$, and $\mathbb{R}_+ = [0, \infty)$. For $a, b \in \mathbb{T}$ with $a < b$, we define the time scales interval by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

LEMMA 2.1. [14, Lemma 2.1] *Let $r : [a, b]_{\mathbb{T}} \rightarrow (0, \infty)$ be a delta differentiable function with $r^{\Delta}(t) \geq 0$ on $[a, b]_{\mathbb{T}}^{\kappa}$. Define*

$$G(x) = \int_{x_0}^x \frac{ds}{g(s)}, \quad x > x_0 > 0, \quad (2.1)$$

where $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ is positive and nondecreasing on $(0, \infty)$. Then, for each $t \in [a, b]_{\mathbb{T}}$, we have

$$G(r(t)) \leq G(r(a)) + \int_a^t \frac{r^{\Delta}(\tau)}{g(r(\tau))} \Delta\tau. \quad (2.2)$$

Throughout this paper, we assume that α is a positive real number with $\alpha \neq 1$, and $\gamma = 1 - \alpha$.

LEMMA 2.2. *Suppose that $y, a \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$. Then*

$$y(t) \leq y_0 + \int_{t_0}^t a(s)y^{\alpha}(s)\Delta s \quad (2.3)$$

for $t \in \mathbb{T}_0$ implies

$$y(t) \leq [y_0^\gamma + \gamma \int_{t_0}^t a(s) \Delta s]^\frac{1}{\gamma}, t \in [t_0, \beta] \cap \mathbb{T}, \quad (2.4)$$

where $\mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}$ and

$$\beta = \sup\{t \in \mathbb{T}_0 : y_0^\gamma + \gamma \int_{t_0}^t a(s) \Delta s > 0\}. \quad (2.5)$$

Proof. Define the function $v(t)$ on \mathbb{T}_0 by

$$v(t) = y_0 + \int_{t_0}^t a(s) y^\alpha(s) \Delta s. \quad (2.6)$$

Then we have

$$\begin{aligned} v^\Delta(t) &= a(t) y^\alpha(t) \\ &\leq a(t) v^\alpha(t) \end{aligned}$$

by the monotonicity of v and

$$\frac{v^\Delta(t)}{v^\alpha(t)} \leq a(t). \quad (2.7)$$

Being the case that $v^\Delta(t) \geq 0$ and $v(t_0) = y_0$, Δ -integrating (2.7) from t_0 to t and applying Lemma 2.1, we obtain

$$G(v(t)) \leq G(v(t_0)) + \int_{t_0}^t a(s) \Delta s, \quad (2.8)$$

where

$$G(x) = \int_{x_0}^x \frac{1}{s^\alpha} ds = \frac{1}{\gamma} [x^\gamma - x_0^\gamma]. \quad (2.9)$$

Thus we have

$$v(t) \leq [y_0^\gamma + \gamma \int_{t_0}^t a(s) \Delta s]^\frac{1}{\gamma}.$$

Since $y(t) \leq v(t)$, we obtain the desired inequality. \square

For our main results, we need the following dynamic inequality which is proved by using Lemma 2.2.

LEMMA 2.3. *Let $y, a, b \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}_+)$, and $-a \in \mathcal{R}^+$. Then*

$$y(t) \leq y_0 + \int_{t_0}^t a(s) y(s) \Delta s + \int_{t_0}^t b(s) y^\alpha(s) \Delta s \quad (2.10)$$

for $t \in \mathbb{T}_0$ implies

$$y(t) \leq \frac{1}{e_{-a}(t, t_0)} [y_0^\gamma + \gamma \int_{t_0}^t b(s) e_{-a}^\gamma(s, t_0) \Delta s]^\frac{1}{\gamma}, \quad t \in [t_0, \beta] \cap \mathbb{T}, \quad (2.11)$$

where

$$\beta = \sup\{t \in \mathbb{T}_0 : y_0^\gamma + \gamma \int_{t_0}^t b(s) e_{-a}^\gamma(s, t_0) \Delta s > 0\}.$$

Proof. Set

$$v(t) = y_0 + \int_{t_0}^t a(s) y(s) \Delta s + \int_{t_0}^t b(s) y^\alpha(s) \Delta s, \quad t \in \mathbb{T}_0.$$

Then we note that $y(t) \leq v(t)$, $v(t_0) = y_0$ and we have

$$\begin{aligned} v^\Delta(t) &= a(t)y(t) + b(t)y^\alpha(t) \\ &\leq a(t)v(t) + b(t)v^\alpha(t) \\ &\leq a(t)v(\sigma(t)) + b(t)v^\alpha(t), \end{aligned} \quad (2.12)$$

since $v(t)$ is nondecreasing in t . It follows from (2.12) that we obtain

$$\begin{aligned} (e_{-a}(t, t_0)v(t))^\Delta &= e_{-a}(t, t_0)v^\Delta(t) - a(t)e_{-a}(t, t_0)v(\sigma(t)) \\ &\leq b(t)e_{-a}(t, t_0)v^\alpha(t) \\ &= b(t)e_{-a}^\gamma(t, t_0)(e_{-a}(s, t_0)v(s))^\alpha, \quad t \in [t_0, \beta] \cap \mathbb{T}. \end{aligned} \quad (2.13)$$

Letting $u(t) = e_{-a}(t, t_0)v(t)$ in (2.13) and then integrating it from t_0 to t , we have

$$u(t) \leq u(t_0) + \int_{t_0}^t b(s) e_{-a}^\gamma(s, t_0) u^\alpha(s) \Delta s. \quad (2.14)$$

It follows from Lemma 2.2 that we have

$$u(t) \leq [u^\gamma(t_0) + \gamma \int_{t_0}^t b(s) e_{-a}^\gamma(s, t_0) \Delta s]^\frac{1}{\gamma}, \quad t \in \mathbb{T}_0, \quad (2.15)$$

which implies that we obtain the desired inequality in (2.11). \square

REMARK 2.4. J. Baoguo et al. obtains a lower bound for $\int_{t_0}^t \frac{u^\Delta(s)}{u^\alpha(s)} \Delta s$ in Corollary 2.5 [4, p. 221] in the following:

Assume that \mathbb{T} satisfies condition C and $u \in C_{\text{rd}}^1(\mathbb{T}, \mathbb{R}_+)$ and $u(t) > 0$ on \mathbb{T}_0 . Then

$$\frac{u^{1-p}(t) - u^{1-p}(t_0)}{1-p} \leq \int_{t_0}^t \frac{u^\Delta(s)}{u^p(s)} \Delta s, \quad t \in \mathbb{T}_0, \quad (2.16)$$

where p is positive and $p \neq 1$.

We can also obtain Lemma 2.3 by using the above lower bound in (2.16).

REMARK 2.5. If we set $\mathbb{T} = \mathbb{R}$ in Lemma 2.3, then we can obtain Theorem 2 in [27] as a continuous version of Lemma 2.3.

Willet and Wong [27, Theorem 4] proved the nonlinear difference inequality by using the mean value theorem. Choi and Koo [12, Theorem 2.3] obtained the inequality which is slightly different from Willet and Wong's Theorem 4.

REMARK 2.6. If we set $\mathbb{T} = \mathbb{Z}$ in Lemma 2.3, then we can obtain Theorem 2.3 in [12] as a discrete version of Lemma 2.3.

If we consider the time scale $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then we obtain the following result.

COROLLARY 2.7. *Let y, a, b be nonnegative real valued sequences defined on $h\mathbb{Z}$. If*

$$y(t) \leq y_0 + \sum_{s=t_0}^{t-h} a(s)y(s)h + \sum_{s=t_0}^{t-h} b(s)y^\alpha(s)h, \quad (2.17)$$

for $t_0 \leq s \leq t$, $t_0, s, t \in h\mathbb{Z}$, then we have

$$y(t) \leq \frac{1}{e_{-a}(t, t_0)} [y_0^\gamma + \gamma h \sum_{s=t_0}^{t-h} b(s)e_{-a}^\gamma]^\frac{1}{\gamma}, t \in [t_0, \beta] \cap h\mathbb{Z} \quad (2.18)$$

where $e_{-a}(t, t_0) = \prod_{s=t_0}^{t-h} (1 - ha(s))$ and

$$\beta = \sup\{t \in h\mathbb{Z} : y_0^\gamma + \gamma \sum_{s=t_0}^{t-h} b(s)e_{-a}^\gamma h > 0\}.$$

We need the following Lemma to prove Theorem 2.9.

LEMMA 2.8. [3, Theorem 2.5.] *Let $t_0 \in \mathbb{T}^\kappa$ and assume $k : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ is continuous at (t, t) , where $t \in \mathbb{T}^\kappa$ with $t > t_0$. Also assume that $k^\Delta(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t , independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$|k(\sigma(t), \tau) - k(s, \tau) - k^{\Delta t}(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad (2.19)$$

for all $s \in U$ where $k^{\Delta t}$ denotes the derivative of k with respect to the first variable. Then

$$g^{\Delta t}(t) = \int_{t_0}^t k^{\Delta t}(t, \tau)\Delta\tau + k(\sigma(t), t), \quad (2.20)$$

where $g(t) = \int_{t_0}^t k(t, \tau) \Delta \tau$.

The following is our main theorem which unify a differential integral inequality in [10, Theorem 2.7] and its corresponding discrete analogue in [12, Theorem 2.5].

THEOREM 2.9. *Suppose that $u, f \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}_+)$, $-f \in \mathcal{R}^+$, and c is a nonnegative constant. Let $k(t, s)$ be defined as in Lemma 2.8 such that $k(\sigma(t), t)$ and $k^{\Delta t}(t, s)$ are rd-continuous functions for $s, t \in \mathbb{T}$ with $s \leq t$. Then*

$$u(t) \leq c + \int_{t_0}^t f(s) [u(s) + \int_{t_0}^s k(s, \tau) u^\alpha(\tau) \Delta \tau] \Delta s, \quad t \in \mathbb{T}_0 \quad (2.21)$$

implies

$$\begin{aligned} u(t) &\leq c + \int_{t_0}^t \frac{f(s)}{e_{-f}(s, t_0)} [c^\gamma \\ &\quad + \gamma \int_{t_0}^s q(\tau) e_{-f}^\gamma(\tau, t_0) \Delta \tau]^\frac{1}{\gamma} \Delta s, \quad t \in [t_0, \beta] \cap \mathbb{T}, \end{aligned} \quad (2.22)$$

where $q(t) = k(\sigma(t), t) + \int_{t_0}^t |k^{\Delta t}(t, s)| \Delta s$ and

$$\beta = \sup\{t \in \mathbb{T}_0 : c^\gamma + \gamma \int_{t_0}^t q(s) e_{-f}^\gamma(s, t_0) \Delta s > 0\}.$$

Proof. Put $v(t)$ by the right hand side of (2.21). Then we have

$$\begin{aligned} v^\Delta(t) &= f(t)u(t) + f(t) \int_{t_0}^t k(t, \tau) u^\alpha(\tau) \Delta \tau, \quad v(t_0) = c \\ &\leq f(t)(v(t) + \int_{t_0}^t k(t, \tau) v^\alpha(\tau) \Delta \tau) \\ &= f(t)m(t), \quad t \in \mathbb{T}_0, \end{aligned} \quad (2.23)$$

where

$$m(t) = v(t) + \int_{t_0}^t k(t, \tau) v^\alpha(\tau) \Delta \tau, \quad m(t_0) = v(t_0) = c.$$

From Lemma 2.8, we obtain

$$\begin{aligned}
m^\Delta(t) &= v^\Delta(t) + k(\sigma(t), t)v^\alpha(t) + \int_{t_0}^t k^{\Delta t}(t, s)v^\alpha(s)\Delta s \\
&\leq f(t)m(t) + k(\sigma(t), t)m^\alpha(t) + \int_{t_0}^t |k^{\Delta t}(t, s)|m^\alpha(s)\Delta s \\
&\leq f(t)m(t) + (k(\sigma(t), t) + \int_{t_0}^t |k^{\Delta t}(t, s)\Delta s|)m^\alpha(t), \quad (2.24)
\end{aligned}$$

since $m(t)$ is nondecreasing in t and $m(t) \geq v(t)$ for each $t \in \mathbb{T}_0$. By Lemma 2.3, we obtain

$$m(t) \leq \frac{1}{e_{-f}(t, t_0)} [m^\gamma(t_0) + \gamma \int_{t_0}^t q(s)e_{-f}^\gamma(s, t_0)\Delta s]^\frac{1}{\gamma}, \quad t \in [t_0, \beta] \cap \mathbb{T}, \quad (2.25)$$

where $q(t) = k(\sigma(t), t) + \int_{t_0}^t |k^{\Delta t}(t, s)|\Delta s$ and

$$\beta = \sup\{t \in \mathbb{T}_0 : m^\gamma(t_0) + \gamma \int_{t_0}^t q(s)e_{-f}^\gamma(s, t_0)\Delta s > 0\}.$$

By substituting (2.25) into (2.23) and then integrating it from t_0 to t , we have

$$v(t) \leq c + \int_{t_0}^t \frac{f(s)}{e_{-f}(s, t_0)} [c^\gamma + \gamma \int_{t_0}^s q(\tau)e_{-f}^\gamma(\tau, t_0)\Delta \tau]^\frac{1}{\gamma} \Delta s$$

for $t \in [t_0, \beta] \cap \mathbb{T}$. Hence the proof is complete. \square

REMARK 2.10. If we suppose further that $k(\sigma(t), t)$ and $k^{\Delta t}(t, s)$ are nonnegative functions for $s, t \in \mathbb{T}$ with $s \leq t$ in Theorem 2.9, then we have

$$\begin{aligned}
u(t) &\leq c + \int_{t_0}^t \left\{ \frac{f(s)}{e_{-f}(s, t_0)} [c^\gamma + \gamma \int_{t_0}^s (k(\sigma(\tau), \tau) \right. \\
&\quad \left. + \int_{t_0}^\tau k^{\Delta \tau}(\tau, \mu)\Delta \mu)e_{-f}^\gamma(\tau, t_0)\Delta \tau]^\frac{1}{\gamma} \right\} \Delta s \quad (2.26)
\end{aligned}$$

for $t \in [t_0, \beta] \cap \mathbb{T}$.

REMARK 2.11. The result of Theorem 2.9 provide an explicit bound for a nonlinear integral inequality of Bellman-Bihari type when $a(t) = c$, $W(v) = v$ and $\Phi(u) = u^\alpha$ in Theorem 2.2 ([14]).

Letting $k(t, s) = h(t)g(s)$ in Theorem 2.9, we obtain the following corollary [23, Theorem 2].

COROLLARY 2.12. Suppose that $u, f, h, g \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}_+)$, $-f \in \mathcal{R}^+$, and c is a nonnegative constant. Suppose that $h^\Delta(t)$ exists, and is an rd-continuous function. Then

$$u(t) \leq c + \int_{t_0}^t f(s)[u(s) + h(s) \int_{t_0}^s g(\tau)u^\alpha(\tau)\Delta\tau]\Delta s \quad (2.27)$$

for all $t \in \mathbb{T}_0$ implies

$$u(t) \leq c + \int_{t_0}^t \frac{f(s)}{e_{-f}(s, t_0)} [c^\gamma + \gamma \int_{t_0}^s (h(\sigma(\tau))g(\tau) + |h^\Delta(\tau)| \int_{t_0}^\tau |g(\sigma)|\Delta\sigma)e_{-f}^\gamma(\tau, t_0)\Delta\tau]^\frac{1}{\gamma} \Delta s, t \in [t_0, \beta] \cap \mathbb{T}, \quad (2.28)$$

where

$$\begin{aligned} \beta &= \sup\{t \in \mathbb{T}_0 \mid c^\gamma + \gamma \int_{t_0}^t (h(\sigma(s))g(s) \\ &\quad + |h^\Delta(s)| \int_{t_0}^s |g(\sigma)|\Delta\sigma)e_{-f}^\gamma(s, t_0)\Delta s > 0\}. \end{aligned}$$

REMARK 2.13. If we set $\mathbb{T} = \mathbb{R}$ in Theorem 2.9, then we can obtain Theorem 2.7 in [10] as a continuous version of Theorem 2.9.

REMARK 2.14. If we set $\mathbb{T} = \mathbb{Z}$ in Theorem 2.9, then we can obtain Theorem 2.4 in [12] as a discrete version of Theorem 2.9.

If we set $k(t, s) = c(t)d(s)$ in Theorem 2.9, then we can obtain the following integral inequality on time scales with a separable kernel from Theorem 2.9. This is an unification of a continuous inequality in [10, Corollary 2.8] and its corresponding discrete analogue in [12, Corollary 2.7].

COROLLARY 2.15. Suppose that $u, f, h, g \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}_+)$, $-f \in \mathcal{R}^+$, and c is a nonnegative constant. Suppose that $h^\Delta(t)$ exists, and is an rd-continuous function for $s, t \in \mathbb{T}$ with $s \leq t$. Then

$$u(t) \leq c + \int_{t_0}^t f(s)[u(s) + h(s) \int_{t_0}^s g(\tau)u^\alpha(\tau)\Delta\tau]\Delta s \quad (2.29)$$

for $t \in \mathbb{T}_0$ implies

$$\begin{aligned} u(t) &\leq c + \int_{t_0}^t \frac{f(s)}{e_{-f}(s, t_0)} [c^\gamma \\ &\quad + \gamma \int_{t_0}^s q(\tau)e_{-f}^\gamma(\tau, t_0)\Delta\tau]^\frac{1}{\gamma} \Delta s, t \in [t_0, \beta] \cap \mathbb{T}, \quad (2.30) \end{aligned}$$

where $q(t) = h(\sigma(t))g(t) + |h^\Delta(t)| \int_{t_0}^t g(s)\Delta s$ and

$$\beta = \sup\{t \in \mathbb{T}_0 : c^\gamma + \gamma \int_{t_0}^t q(s)e_{-\gamma}^\gamma(s, t_0)\Delta s > 0\}.$$

If we set $\alpha = 1$ in Theorem 2.9, then we can obtain the following corollary which is proved in Theorem 3.4[11].

COROLLARY 2.16. *Let $u, f \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}_+)$, and c be a nonnegative constant. Suppose that $k(t, s)$ is defined as in Lemma 2.8 such that $k(\sigma(t), t)$ and $k^{\Delta t}(t, s)$ are rd-continuous functions for $s, t \in \mathbb{T}$ with $s \leq t$.*

Then

$$u(t) \leq c + \int_{t_0}^t f(s)[u(s) + \int_{t_0}^s k(s, \tau)u(\tau)\Delta\tau]\Delta s \quad (2.31)$$

for all $t \in \mathbb{T}_0$ implies

$$u(t) \leq c[1 + \int_{t_0}^t f(s)e_p(s, t_0)\Delta s], \quad t \in \mathbb{T}_0, \quad (2.32)$$

where $p(t) = f(t) + k(\sigma(t), t) + \int_{t_0}^t k^{\Delta t}(t, s)\Delta s$.

The proof of this corollary follows by the similar argument as in the proof of Theorem 2.9. We omit the details.

REMARK 2.17. If we set $p(t) = 0$ in [19, Theorems 3.1 and 3.2], then we easily obtain Pachpatte type inequalities of Theorems 3.1 and 3.2 in [19] as a corollary of Theorem 2.9. Furthermore, if we set $k(t, s) = d(s)$ in Corollary 2.16, then we have

$$\begin{aligned} p(t) &= f(t) + k(\sigma(t), t) + \int_{t_0}^t k^{\Delta t}(t, s)\Delta s \\ &= f(t) + d(t). \end{aligned}$$

Thus we also obtain Theorem 1 in [28] from Corollary 2.16.

Another variant of Theorem 2.9 is given by the following theorem. If we use Lemma 2.3 in the proof of Theorem 2.9, then we can obtain the following bound of $u(t)$ which is different from Theorem 2.9.

THEOREM 2.18. *Suppose that $u, f \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}_+)$, $-f \in \mathcal{R}^+$, and c is a nonnegative constant. Let $k(t, s)$ be defined as in Lemma 2.8. Then*

$$u(t) \leq c + \int_{t_0}^t f(s)[u(s) + \int_{t_0}^s k(s, \tau)u^\alpha(\tau)\Delta\tau]\Delta s \quad (2.33)$$

for $t \in \mathbb{T}_0$ implies

$$u(t) \leq \frac{1}{e_{-f}(t, t_0)} [c^\gamma + \gamma \int_{t_0}^t f(s) (\int_{t_0}^s k(s, \tau) \Delta\tau) e_{-f}^\gamma(s, t_0) \Delta s]^\frac{1}{\gamma} \quad (2.34)$$

for $t \in [t_0, \beta] \cap \mathbb{T}$ where

$$\beta = \sup\{t \in \mathbb{T}_0 : c^\gamma + \gamma \int_{t_0}^t f(s) (\int_{t_0}^s k(s, \tau) \Delta\tau) e_{-f}^\gamma(s, t_0) \Delta s > 0\}.$$

Proof. Define a function $v(t)$ by the right hand side of (2.33). Then, we have

$$\begin{aligned} v^\Delta(t) &= f(t)u(t) + f(t) \int_{t_0}^t k(t, \tau) u^\alpha(\tau) \Delta\tau, \quad v(t_0) = c \\ &\leq f(t)v(t) + (f(t) \int_{t_0}^t k(t, \tau) \Delta\tau) v^\alpha(t), \quad t \in \mathbb{T}_0, \end{aligned}$$

since $u(t) \leq v(t)$ and $v^\alpha(t)$ is nondecreasing in t . From Lemma 2.3, we obtain

$$v(t) \leq \frac{1}{e_{-f}(t, t_0)} [c^\gamma + \gamma \int_{t_0}^t f(s) (\int_{t_0}^s k(s, \tau) \Delta\tau) e_{-f}^\gamma(s, t_0) \Delta s]^\frac{1}{\gamma}$$

for $t \in [t_0, \beta] \cap \mathbb{T}$ where

$$\beta = \sup\{t \in \mathbb{T}_0 : c^\gamma + \gamma \int_{t_0}^t f(s) (\int_{t_0}^s k(s, \tau) \Delta\tau) e_{-f}^\gamma(s, t_0) \Delta s > 0\}.$$

Since $u(t) \leq v(t)$, the proof is complete. \square

If we set $\mathbb{T} = \mathbb{Z}$ in Theorem 2.18, then we can obtain the following result as a discrete version of Theorem 2.18. Let $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots, n_0 + k, \dots\}$ and $\mathbb{N}(n_0, l) = \{n_0, n_0 + 1, \dots, n_0 + k, \dots, l\}$ for fixed nonnegative integers n_0 and l .

COROLLARY 2.19. *Let $u(n)$ and $f(n) < 1$ be nonnegative sequences defined on $\mathbb{N}(n_0)$ and $k(n, m)$ be a nonnegative function for $n, m \in \mathbb{N}(n_0)$ with $n \geq m$. Suppose that*

$$u(n) \leq c + \sum_{s=n_0}^{n-1} f(s) [u(s) + \sum_{\tau=n_0}^{s-1} k(s, \tau) u^\alpha(\tau)], \quad n \in \mathbb{N}(n_0), \quad (2.35)$$

where c is a positive constant. Then we have

$$u(n) \leq \frac{1}{e_{-f}} [c^\gamma + \gamma \sum_{s=n_0}^{n-1} (f(s) \sum_{\tau=n_0}^{s-1} k(s, \tau)) e_{-f}^\gamma(s)]^\frac{1}{\gamma} \quad (2.36)$$

for $n \in \mathbb{N}(n_0, \beta)$ where $e_{-f}(n) = \prod_{s=n_0}^{n-1} (1 - f(s))$ and

$$\beta = \sup\{n \in \mathbb{N}(n_0) : c^\gamma + \gamma \sum_{s=n_0}^{n-1} (f(s) \sum_{\tau=n_0}^{s-1} k(s, \tau)) e_{-f}^\gamma(s) > 0\}.$$

3. Examples

In this section, we use Lemma 2.3 and Theorem 2.9 in order to study the qualitative analysis of nonlinear dynamic equations. Let $t_0 \in \mathbb{T}$ and consider the initial value problem

$$u^\Delta(t) = F(t, u(t), \int_{t_0}^t K(t, s, u(s)) \Delta s), t \in \mathbb{T}^\kappa, u(t_0) = u_0, \quad (3.1)$$

where $u \in C_{\text{rd}}^1(\mathbb{T}, \mathbb{R})$, $F \in C_{\text{rd}}(\mathbb{T} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $K \in C_{\text{rd}}(\mathbb{T} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In what follows, we shall assume that the IVP (3.1) has a unique solution, which we denote by $u(t)$.

EXAMPLE 3.1. Assume that the functions F and K in (3.1) satisfy the following conditions

$$|K(t, s, u)| \leq k(t, s)|u|^\alpha, \quad (3.2)$$

$$|F(t, u, v)| \leq a(t)(|u| + |v|), \quad (3.3)$$

where two functions k and a are defined in Theorem 2.9. Then we have

$$\begin{aligned} |u(t)| &\leq c + \int_{t_0}^t \frac{a(s)}{e_{-a}(s, t_0)} [c^\gamma \\ &+ \gamma \int_{t_0}^s q(\tau) e_{-a}^\gamma(\tau, t_0) \Delta \tau]^\frac{1}{\gamma} \Delta s, t \in [t_0, \beta] \cap \mathbb{T}, \end{aligned} \quad (3.4)$$

where $q(t) = k(\sigma(t), t) + \int_{t_0}^t |k^{\Delta t}(t, s)| \Delta s$ and

$$\beta = \sup\{t \in \mathbb{T}_0 : c^\gamma + \gamma \int_{t_0}^t q(s) e_{-a}^\gamma(s, t_0) \Delta s > 0\}.$$

If $\mathbb{T} = \mathbb{R}$, then we can obtain a bound for the solution of the initial value problem by using the integral inequality of Bernoulli-type.

EXAMPLE 3.2. Consider the nonlinear differential equation with the initial condition $x(0)$

$$x' = -\frac{x}{2 + \sin x} + \frac{tx^4}{1 + x^2}, t \geq 0. \quad (3.5)$$

If $x(t)$ is a solution of (3.5) satisfying $x(0) = 1$, then we have

$$|x(t)| \leq \frac{e^t}{[\frac{2}{3} + e^{3t}(\frac{1}{3} - t)]^{\frac{1}{3}}}, t \in [0, \beta), \quad (3.6)$$

where

$$\beta = \sup\{t : \frac{2}{3} + e^{3t}(\frac{1}{3} - t) > 0\}.$$

Proof. The initial value problem (3.5) is equivalent to the integral equation

$$x(t) = 1 - \int_0^t \frac{x(s)}{2 + \sin x(s)} ds + \int_0^t \frac{sx^4(s)}{1 + x^2(s)} ds, \quad t \geq 0. \quad (3.7)$$

Putting $u(t) = |x(t)|$, we obtain

$$u(t) \leq 1 + \int_0^t u(s) ds + \int_0^t su^4(s) ds, \quad t \geq 0. \quad (3.8)$$

When $u_0 = 1, a = 1, b(t) = t$ and $\alpha = 4$ in Lemma 2.3, we obtain

$$\begin{aligned} u(t) &\leq \frac{e^t}{[1 - 3 \int_0^t se^{3s} ds]^{\frac{1}{3}}} \\ &= \frac{e^t}{[\frac{2}{3} + e^{3t}(\frac{1}{3} - t)]^{\frac{1}{3}}}, t \in [0, \beta), \end{aligned}$$

where

$$\beta = \sup\{t \in \mathbb{R}_+ : \frac{2}{3} + e^{3t}(\frac{1}{3} - t) > 0\}.$$

□

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